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WEAK ISOMORPHISM OF MEASURE-PRESERVING DIFFEOMORPHISMS

BY

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ABSTRACT

On the two-dimensional torus we construct two C^∞-diffeomorphisms T_1, T_2 satisfying:

(i) T_1, T_2 preserve Lebesgue measure and are ergodic with respect to it,

(ii) T_1, T_2 are measurable factors of each other,

(iii) T_1, T_2 are not measure-theoretically isomorphic.

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Introduction

The weak isomorphism problem in ergodic theory has a long and interesting history. In 1963, Sinai [29] asked whether it was possible to find an ergodic automorphism T such that:

(1) T has a weakly isomorphic factor which is not isomorphic to T.

The famous result of Ornstein [20], [21] saying that two Bernoulli shifts are isomorphic iff they have the same entropy, shows that (1) could not hold for T a Bernoulli shift. In 1968, Hahn and Parry [6] introduced the notion of the coalescence of an automorphism. If T is coalescent (i.e. if each measure-preserving transformation commuting with T is invertible) once more (1) cannot hold for it. In particular, all automorphisms with quasi-discrete spectrum and those with no spectral type of infinite spectral multiplicity ([1], [18]) are coalescent (these are examples of automorphisms of zero entropy).

In 1974, Polit [25] constructed a zero-entropy mixing example satisfying (1). Later, one of the authors (Rudolph [26]) generalizing on Polit's work introduced the notion of minimal self-joinings (MSJ) of an automorphism. It is known that if T enjoys MSJ then the infinite direct product automorphism

(2) $$T \times T \times \cdots \quad \text{satisfies (1).}$$

In [10], [12], [26] various examples of automorphisms with the MSJ property (hence, of zero entropy) have been constructed. In [13], the authors noticed that a weaker property of T, called simplicity, was enough to get (2). In 1986, Thouvenot [30] proved that if T was a Gaussian automorphism with simple spectrum then (2) holds true. Recently, in [11], the authors have shown that property (2) is a "typical" (with respect to the weak topology [7]) property of automorphisms of a Lebesgue space.

The examples of automorphisms satisfying (1) presented above are all weakly mixing, however, it is very likely that no one of them enjoys the loosely Bernoulli (LB) property (a zero entropy automorphism is LB iff it is induced from an irrational rotation [22]). In [4], [15], [16] some examples of LB automorphisms satisfying (1) have been constructed.

However, all the examples of ergodic automorphisms for which (1) is satisfied were constructed as some (essentially) infinite self-joining (see [3]). Hence, a natural question arises whether or not we can construct an ergodic automorphism satisfying (1) which is not an (essentially) infinite self-joining of another automorphism. In [3], it was noticed that if $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ is an irrational rotation and $\varphi\colon X \longrightarrow X$ is a measurable map then whenever the automorphism

$$(3) \qquad T_\varphi\colon (X\times X,\mu\times\mu) \longrightarrow (X\times X,\mu\times\mu),\ \ T_\varphi(x,y) = (Tx,\varphi(x)\cdot y)$$

is ergodic, it cannot be isomorphic to an (essentially) infinite self-joining. The automorphisms of the two-dimensional torus of the form (3) will be called Anzai skew products. In [17], an ergodic Anzai skew product without the coalescence property was constructed. Although, using the same methods, an improvement of the construction from [17] to obtain the stronger (1) property of T_φ is possible, we do not go in this direction any further since, as noticed in [3], in the cohomology class of the φ from [17] there is no absolutely continuous cocycle. (We recall, that in the cohomology class of an arbitrary cocycle there is one which is continuous [14], [27].) In particular, no coboundary modification of the φ could lead to a diffeomorphism of the two-dimensional torus.

The main problem we deal with in this paper is a construction of an ergodic diffeomorphism preserving a smooth measure on a finite dimensional compact smooth manifold and satisfying (1). Such a construction is impossible on the circle as from Denjoy's theorem each C^1-diffeomorphism on the circle without periodic points and with derivative of bounded variation is strictly ergodic and isomorphic to a rotation. In contrast to this, on the two-dimensional torus, we will construct two ergodic C^∞-diffeomorphisms (preserving Lebesgue measure) which are weakly isomorphic but not isomorphic. In Section 5 we deal with C^∞-diffeomorphisms of the form (3) proving that their C^∞-centralizer is uncountable.

Some open questions are listed at the end of the paper.

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1. Definitions and Notation

Let $(X,\mathcal{B},\mu)$ be a Lebesgue space with a normalized measure μ. Assume that $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ is an invertible measure-preserving transformation (*i.e.* T is an **automorphism**). Let $C(T)$ denote the **centralizer** of T which is the set of all not necessarily invertible measure-preserving transformations commuting with T. Assume that $\mathcal{A} \subset \mathcal{B}$ is a T-invariant sub-σ-algebra. Then, the quotient action of T on $\mathcal{A}$ is called a **factor** of T (we will identify factors with T-invariant sub-σ-algebras).

Let G be a compact abelian metric group with Haar measure m. A measurable function $\bar{\varphi}: \mathbf{Z} \times X \longrightarrow G$ is called a **cocycle** if $\bar{\varphi}^{(n+k)}(x) = \bar{\varphi}^{(n)}(x) \cdot \bar{\varphi}^{(k)}(T^n(x))$. Any such is clearly of the form $\bar{\varphi}^{(n)}(x) = \prod_{j=0}^{n-1} \varphi(T^j(x))$, $n \geq 0$, $\bar{\varphi}^{(n)}(x) = (\prod_{j=n}^{-1} \varphi(T^j(x)))^{-1}$, $n < 0$, where $\varphi(x) = \bar{\varphi}(1, x)$ is the "generator" of the cocycle. Abusing language we will refer to φ as "the" cocycle although we will be referring to the cocycle it generates. A cocycle φ determines an automorphism T_φ (called a **G-extension** of T) on $(X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ by

$$T_\varphi(x, g) = (Tx, g \cdot \varphi(x)), \tag{4}$$

($\tilde{\mathcal{B}}$ is the product σ -algebra and $\tilde{\mu} = \mu \times m$). A cocycle φ is said to be a **coboundary** (or a **G-coboundary** if we need to emphasize the role of G) if it is of the form

$$\varphi(x) = f(Tx)/f(x)$$

for a measurable function $f: X \longrightarrow G$. We say that two cocycles $\varphi, \psi: X \longrightarrow G$ are **cohomologous** if φ/ψ is a coboundary. Assume that T is ergodic. We will say that φ is **ergodic** if T_φ is. The following is classical [23].

$$\begin{cases} \varphi \text{ is ergodic iff for no character } \chi \in \hat{G}, \chi \neq 1, \\ \text{the cocycle } \chi \circ \varphi \text{ is an } \mathbf{S}^1\text{-coboundary.} \end{cases} \tag{5}$$

If T_φ: $(X \times G, \tilde{\mu}) \longrightarrow (X \times G, \tilde{\mu})$ is a G-extension of T then it has a system of factors called **natural factors** arising as follows.

Let $H \subset G$ be a compact subgroup of G. Consider

$$\tilde{\mathcal{B}}_H = \{ \tilde{A} \in \tilde{\mathcal{B}}: (\forall h \in H) \ \sigma_h \tilde{A} = \tilde{A} \},$$

where $\sigma_h(x, g) = (x, hg)$. The corresponding factor, denoted by $T_{\varphi,H}$, is simply the action of T_φ on $(X \times G/H, \tilde{\mathcal{B}}_H, \tilde{\mu})$. When $G = \mathbf{S}^1$ such H are either finite or all of $\mathbf{S}^1$.

PROPOSITION 1 ([13]): *Assume that T_φ is an ergodic G-extension of T and let $\mathcal{A}$ be a factor of T_φ such that*

$$\{B \times G :: B \in \mathcal{B}\} \subset \mathcal{A}.$$

Then $\mathcal{A}$ is a natural factor of T_φ.

Let $T: (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic rotation on a compact monothetic group X with Haar measure μ (*i.e.* up to isomorphism T is assumed to have discrete spectrum).

PROPOSITION 2 ([19]): *Suppose that φ, ψ: $X \longrightarrow G$ are ergodic cocycles. T_φ and T_ψ are isomorphic iff there exist $S \in C(T)$, a measurable map f: $X \longrightarrow G$ and a continuous group automorphism v: $G \longrightarrow G$ such that*

$$\varphi \circ S / v(\psi) = f \circ T / f.$$

Remark 1: We will also consider maps φ: $X \longrightarrow \mathbf{R}$. As before we abuse language and call such a function a **cocycle** if it is measurable. We write the actual cocycle as

$$\varphi^{(n)}(x) = \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x)$$

for $n > 0$, $\varphi^{(0)} = 0$. If $n \leq -1$ then

$$\varphi^{(n)}(x) = -\varphi(T^{-1}x) - \cdots - \varphi(T^{n}x). \qquad \blacksquare$$

2. Game plan

From now on we assume that $X = \mathbf{S}^1$ (the circle) and μ is Lebesgue measure. Let $Tx = x \cdot e^{2\pi i\alpha}$ for some irrational $\alpha \in [0,1)$. Assume that $\varphi : X \longrightarrow X$ is an *ergodic* cocycle and let $T_\varphi : (X \times X, \tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow (X \times X, \tilde{\mathcal{B}}, \tilde{\mu})$ be the corresponding Anzai skew product. Suppose that $\mathcal{A} \subset \tilde{\mathcal{B}}$ is a factor that is weakly isomorphic to T_φ. Then

$$\{B \times X \colon B \in \mathcal{B}\} \subset \mathcal{A}$$

since T_φ is ergodic and the σ-algebra $\{B \times X \colon B \in \mathcal{B}\}$ is determined by the eigenfunctions of T_φ. In view of Proposition 1, $\mathcal{A}$ has to be a natural factor. As T_φ is not isomorphic to T, $H \neq \mathbf{S}^1$ and so is finite, *i.e.* the kth roots of unity for some $k \geq 1$. We conclude that the action of T_φ on $\mathcal{A}$ is isomorphic to $T_{\varphi^k} \colon (X \times X, \tilde{\mathcal{B}}, \tilde{\mu}) \longrightarrow (X \times X, \tilde{\mathcal{B}}, \tilde{\mu})$. Notice that if $T_{\varphi^{2k}}$ and T_φ are isomorphic then certainly T_φ and T_{φ^k} are weakly isomorphic and from the discussion above this is the only way to get such an example within the context of Anzai skew products. Therefore, using $k = 2$ to construct two weakly isomorphic Anzai skew products that are not isomorphic we will find α and β from $[0,1)$ and a cocycle $\varphi\colon X \longrightarrow X$ such that if we denote $Tx = x \cdot e^{2\pi i\alpha}$, $Sx = x \cdot e^{2\pi i\beta}$ $(x \in \mathbf{S}^1)$, then

$$T_\varphi \text{ is ergodic,} \tag{6}$$

$$\varphi(Sx)/(\varphi(x))^4 = f(Tx)/f(x) \tag{7}$$

for a measurable function $f\colon X \longrightarrow X$ and moreover

$$\left\{ \begin{array}{l} \text{for an arbitrary } U \in C(T) \text{ there is no measurable solution} \\ g\colon X \longrightarrow X \text{ of the equation } \varphi(Ux)/(\varphi(x))^{\pm 2} = g(Tx)/g(x). \end{array} \right. \tag{8}$$

Indeed, by (6) and (8) and Proposition 2 it follows that T_φ and T_{φ^2} are not isomorphic, while (7) and Proposition 2 state that T_φ and T_{φ^4} are isomorphic.

Suppose that $\varphi\colon X \longrightarrow X$ is continuous. Let $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ be its natural continuous lifting with $\tilde{\varphi}(0) \in [0,1)$. The number $\tilde{\varphi}(1) - \tilde{\varphi}(0) \in \mathbf{Z}$ is called the **degree** $d(\varphi)$ of φ. If $\tilde{\varphi}$ is absolutely continuous and $d(\varphi) \neq 0$ then, by a result of [3], T_φ is coalescent and (1) fails to be true. Hence, we have to consider the case of $d(\varphi) = 0$, in other words we study continuous maps $\tilde{\varphi}\colon X \longrightarrow \mathbf{R}$ (or which is the same $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is periodic of period 1). For such a case, in order to solve (7), it is enough to find a measurable solution $\tilde{f}\colon \mathbf{R} \longrightarrow \mathbf{R}$ periodic of period 1, to the equation

$$\tilde{\varphi}(x+\beta) - 4\tilde{\varphi}(x) = \tilde{f}(x+\alpha) - \tilde{f}(x). \tag{9}$$

By taking the exponentials in (9) we get (7). We will call $\tilde{f}$ in (9) a **transfer** function.

Remark 2: Transfer functions are rather "wild" functions. This can be made precise by the following observation.

Let $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ be an ergodic automorphism. Assume that $S \in C(T)$ is invertible. Moreover, assume that $\tilde{\varphi}\colon X \longrightarrow \mathbf{R}$ is a cocycle, $\tilde{\varphi} \in L^1_{\mathbf{R}}(X,\mu)$ and $\tilde{\varphi}$ is not an $\mathbf{R}$-coboundary. If there exists a measurable $\tilde{f}\colon X \longrightarrow \mathbf{R}$ satisfying

$$\tilde{\varphi}(Sx) - k\tilde{\varphi}(x) = \tilde{f}(Tx) - \tilde{f}(x)$$

$(x \in X)$ for some $k \in \mathbf{Z}, |k| > 1$ then $\tilde{f} \notin L^1_{\mathbf{R}}(X,\mu)$.

Indeed, let us define an operator $A_{S,k}\colon L^1_{\mathbf{R}}(X,\mu) \longrightarrow L^1_{\mathbf{R}}(X,\mu)$ by putting $A_{S,k}(g) = g \circ S - k \cdot g$. This operator is linear and continuous. We will show that it is invertible. Put $L\colon L^1_{\mathbf{R}}(X,\mu) \longrightarrow L^1_{\mathbf{R}}(X,\mu)$, $L(g) = k \cdot g \circ S^{-1}$. We have

$||L^{-n}|| = 1/|k|^n$, $n \geq 1$, so $M = \sum_{n=1}^{\infty} L^{-n}$ is well defined and $(Id-L)\circ M = Id$. Thus $Id - L$ is invertible and so also is $A_{S,k} = (Id - L) \circ S$.

Suppose, now, that $f \in L^1_{\mathbf{R}}(X, \mu)$. Then, since $S \in C(T)$,

$$\tilde{\varphi} = A_{S,k}^{-1}(\tilde{f} \circ T - \tilde{f}) = (A_{S,k}^{-1}\tilde{f}) \circ T - A_{S,k}^{-1}(\tilde{f})$$

and $\tilde{\varphi}$ is a coboundary, a contradiction. ∎

We will have to show that the left side cocycle in (9) is an **R**-coboundary. We will use the following rather standard sort of criterion ([28]).

PROPOSITION 3: *Let $T\colon (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism. Assume that $\tilde{\psi}\colon X \longrightarrow \mathbf{R}$ is a cocycle. Then $\tilde{\psi} = \tilde{f} \circ T - \tilde{f}$ for a measurable $\tilde{f}\colon X \longrightarrow \mathbf{R}$ iff there exists a set $S \subset X$, $\mu(S) > 0$, such that*

$$|\tilde{\psi}^{(k)}(x)| \leq 1 \tag{10}$$

whenever $x, T^k x \in S$ and $k \in \mathbf{Z}$.

Proof: Suppose that $\tilde{f}\colon X \longrightarrow \mathbf{R}$ is measurable and

$$\tilde{f}(Tx) - \tilde{f}(x) = \tilde{\psi}(x). \tag{11}$$

Since $\tilde{f}$ is measurable, there is a real number r such that the set

$$S = \{x \in X\colon |\tilde{f}(x) - r| < 1/2\}$$

has positive measure. It follows from (11) that $\tilde{\psi}^{(k)}(x) = \tilde{f}(T^k x) - \tilde{f}(x)$, $k \in \mathbf{Z}$, $x \in X$. Now, if $x, T^k x \in S$ then $|\tilde{\psi}^{(k)}(x)| \leq |\tilde{f}(T^k x) - r| + |\tilde{f}(x) - r| < 1$.

Suppose, now, that (10) is satisfied for a set S of positive measure. The ergodicity of T implies that for a.a. $x \in X$ there is $k = k(x) \in \mathbf{Z}$ with $T^k x \in S$. Therefore the set

$$A(x) = \{\tilde{\psi}^{(k)}(x)\colon T^k x \in S,\ k \in \mathbf{Z}\}$$

is nonempty. Moreover, it is bounded above.

Indeed, choose $k_0 \in \mathbf{Z}$ with $T^{k_0} x \in S$. Let k be any integer, $k \neq k_0$, and $T^k x \in S$. Then

$$\tilde{\psi}^{(k)}(x) = \tilde{\psi}^{(k_0)}(x) + \tilde{\psi}^{(k-k_0)}(T^{k_0} x) \leq \tilde{\psi}^{(k_0)}(x) + 1,$$

as $T^{k_0} x, T^{k-k_0}(T^{k_0} x) \in S$ and (10) holds true.

Let us define a function $\tilde{f}\colon X \longrightarrow \mathbf{R}$ by

$$\tilde{f}(x) = -\sup A(x).$$

Hence, $\tilde{f}$ is finite for a.e. $x \in X$ and is measurable since

$$\{x\colon \tilde{f}(x) \le -b\} = \bigcup_{k=-\infty}^{\infty} [\{x\colon \tilde{\psi}^{(k)}(x) \ge b\} \cap T^{-k}(S)]$$

for every $b \in \mathbf{R}$. Since $A(Tx) = A(x) - \tilde{\psi}(x)$, we have $\tilde{f}(Tx) - \tilde{f}(x) = \tilde{\psi}(x)$.

In applications, to show that $\tilde{\psi}$ is a coboundary it is enough to have a set S of positive measure with the property that

$$|\tilde{\psi}^{(k)}|(x) \le 1 \quad \text{whenever } x, T^k x \in S \quad \text{for all } k \ge 1.$$

Indeed, if $k = 0$ then (10) holds true. Suppose that now $k < 0$ and $x, T^k x \in S$. Then $T^k x, T^{-k}(T^k x) \in S$ with $-k > 0$ so $|\tilde{\psi}^{(-k)}(T^k x)| \le 1$ and since $|\tilde{\psi}^{(-k)}(T^k x)| = |\tilde{\psi}^{(k)}(x)|$, we are done.

COROLLARY 1: *Assume that $\xi = \{F, TF, \ldots, T^{n-1}F\}$ is a Rokhlin tower for T, $\mu(F) > 0$. Let $\tilde{\varphi}\colon X \longrightarrow \mathbf{R}$ be a cocycle satisfying*

$$\tilde{\varphi} = 0 \ \text{on} \ X \setminus \bigcup_{i=0}^{n-1} T^i F, \tag{12}$$

$$\sum_{i=0}^{n-1} \tilde{\varphi}(T^i x) = 0 \ \text{for} \ x \in F. \tag{13}$$

then $\tilde{\varphi}$ is an $\mathbf{R}$-coboundary.

Proof: It suffices to apply Proposition 3 with $S = F$. ∎

COROLLARY 2: *Let $\tilde{\varphi}_i\colon X \longrightarrow \mathbf{R}$ be a cocycle such that there exists a set S_i with $\mu(S_i) > 1 - \varepsilon_0/2^i$ for some $\varepsilon_0 < 1$ and $x, T^k x \in S_i$ implies $|\tilde{\varphi}^{(k)}(x)| < 1/2^i$, $k \ge 1$, $i = 1, 2, \ldots$.*
If the series $\sum_{i=1}^{\infty} \tilde{\varphi}_i(x)$ is convergent a.e. then the cocycle $\tilde{\varphi}(x) = \sum_{i=1}^{\infty} \tilde{\varphi}_i(x)$ is a coboundary.

Proof: Set $S = \bigcap_{i=1}^{\infty} S_i$ and observe that $\mu(S) \ge 1 - \varepsilon_0 > 0$. Now, if $x, T^k(x) \in S$ then $x, T^k(x) \in S_i$ for every i and therefore $|\tilde{\varphi}_i^{(k)}(x)| < 1/2^i$ which implies $|\tilde{\varphi}^{(k)}(x)| < 1$. ∎

Remark 3: Gottschalk and Hedlund in [5] have proved that if $T\colon X \longrightarrow X$ is a minimal homeomorphism of a compact metric space and if $\tilde{\varphi}\colon X \longrightarrow \mathbf{R}$ is a continuous map then $\tilde{\varphi} = \tilde{f} \circ T - \tilde{f}$ for a continuous $\tilde{f}\colon X \longrightarrow \mathbf{R}$ iff there exists a point $x_0 \in X$ such that $\{\tilde{\varphi}^{(k)}(x_0)\colon k \geq 0\}$ is bounded. Proposition 3 is an analogue of that theorem in the measure-theoretic case. ∎

Given an irrational $\alpha \in [0,1)$ and hence $T\colon X \longrightarrow X$, where $Tx = x \cdot e^{2\pi i \alpha}$ we seek β and $\tilde{\varphi}\colon X \longrightarrow \mathbf{R}$ with (6),(8) and (9). Certainly, this is not possible for an arbitrary α if we want φ to have a smooth coboundary modification. For instance, for irrationals with bounded partial quotients such a $\tilde{\varphi}$ has to be cohomologous to a constant (classical small divisor arguments). We will deal with α's satisfying the following condition.

Denote $\tilde{T}(x) = x + \alpha \pmod 1$, where $x \in [0,1)$. Let $\alpha = [0\colon a_1, a_2, \ldots)$ be the continued fraction expansion of α with the convergents P_n/Q_n. Assume that some sequences $\{\varepsilon_k\}$ with

$$0 < \varepsilon_k < 1/(10 \cdot 2^k) \tag{14}$$

and $\{C_k\}$, with $C_k > 0$, are given.

Definition 1: We say that $\alpha \in [0,1)$ satisfies the (R) condition with respect to $\{\varepsilon_k\}$ and $\{C_k\}$ if there exists a subsequence $\{n_k\}$ of natural numbers such that

(R1) $\sum_{k=1}^{\infty} \frac{Q_{2n_k}^k}{a_{2n_k+1}} \cdot C_k < +\infty$,

(R2) $\frac{2}{a_{2n_k+1}} < \varepsilon_k, \quad k = 1, 2, \ldots.$

If, besides,

(R3) $a_{2n_k+1} = 2p(k)q(k)$, where $p(k) > \frac{1}{4\varepsilon_k}$, $q(k) \nearrow \infty$,

(R4) $Q_{2n_k} \geq 1/\varepsilon_k^2$,

then we say that α satisfies the full *(R)* condition. ∎

Theorem 1: *The set of α's satisfying the full* (R) *condition with respect to $\{\varepsilon_k\}$, $\{C_k\}$ is residual.*

The proof of Theorem 1 and some more facts concerning properties of the continued fraction expansion of α are postponed to the appendix.

Suppose, now, that $\alpha \in [0,1)$ satisfies (R2) and (R4) along a subsequence $\{n_k\}$. Denote

$$r(k) = a_{2n_k+1}, \quad h_k = Q_{2n_k}$$

$$J_1^k = [0, Q_{2n_k}\alpha \pmod 1), \quad I_k = [0, a_{2n_k+1} \cdot Q_{2n_k}\alpha \pmod 1), \quad k \geq 1.$$

Then, we have

$$\xi_k = \{I_k, \tilde{T}I_k, \ldots, \tilde{T}^{h_k-1}I_k\} \text{ is a Rokhlin tower ,} \tag{15}$$

$$\mu(\bigcup_{i=0}^{h_k-1} \tilde{T}^i I_k) > 1 - \varepsilon_k, \quad h_k \geq 1/\varepsilon_k^2, \tag{16}$$

$$\left\{ \begin{array}{l} I_k \text{ is the disjoint union of intervals } J_s^k,\ s = 1,2,\ldots,r(k), \text{ listed left} \\ \text{to right within } I_k \text{ such that } \tilde{T}^{h_k}(J_s^k) = J_{s+1}^k,\ s = 1,2,\ldots,r(k)-1, \end{array} \right. \tag{17}$$

(this is just to say that $\tilde{T}^{h_k}$ translates I_k by the length of J_1^k)

$$\left\{ \begin{array}{l} I_{k+1} \subset J_1^k,\ k \geq 1, \text{ and whenever an interval from } \xi_{k+1} \text{ is contained} \\ \text{in } J_1^k, \text{ its } r(k)h_k - 1 \text{ iterations are also in } \xi_{k+1}. \end{array} \right. \tag{18}$$

The proofs of (15) – (18) can be found in the appendix.

Remark 4: Each α satisfying the (R) condition has to be irrational (from (R2), the a_i are clearly unbounded). ∎

Given α satisfying (R2),(R3) and (R4) the number β will be constructed as the intersection of a decreasing sequence of closed intervals. We will construct a cocycle $\tilde{\varphi}$: $\mathbf{R} \longrightarrow \mathbf{R}$ (periodic of period 1) as

$$\tilde{\varphi}(x) = \sum_{i=1}^{\infty} \tilde{\varphi}_i(x), \quad x \in [0,1),$$

where $\tilde{\varphi}_k$ will be nonzero only on $\bigcup_{j=2}^{r(k)} J_j^k$ and will be constant on each J_j^k. In particular, the $\tilde{\varphi}$ constructed is not even continuous. We will show that for some appropriate choice of the values of $\tilde{\varphi}_i$ we can reach a cocycle $\tilde{\varphi}$ satisfying (6),(8) and (9). The cocycles $\tilde{\varphi}_i$ will be $\mathbf{R}$-coboundaries, while $\tilde{\varphi}$ will not, actually (6) will hold. Consequently, $\tilde{\psi}_i = \tilde{\varphi}_i S - 4\tilde{\varphi}_i$ are all coboundaries with the corresponding (from Proposition 3) set S_i. We will be sure the sets S_i satisfy the assumptions of Corollary 2 with $\tilde{\psi}_i^{(k)}(x) = 0$, whenever $x, \tilde{T}^k x \in S_i$, $k \geq 1$. Some combinatorial conditions on the values of $\tilde{\varphi}_i$ will force (8) (and (6)) to hold. Finally, we will show that under the assumption that α satisfies the (R) condition, $\tilde{\varphi}_i$ has a coboundary modification to a C^∞-cocycle $\tilde{\eta}_i$ in such a way that the cocycle $\tilde{\eta} = \sum_{i=1}^{\infty} \tilde{\eta}_i$ is a C^∞-map cohomologous to $\tilde{\varphi}$.

3. Construction of weakly isomorphic Anzai skew products that are not isomorphic

We require α to satisfy (R2), (R3) and (R4), so the integers $r(k), k \geq 1$, fulfil the following condition:

$$r(k) = 8p(k)q(k), \text{ where } q(k) > 1,\ q(k) \nearrow \infty \text{ and } 1/p(k) < \frac{1}{4}\varepsilon_k. \tag{19}$$

Let us denote

$$\tilde{T}^i(J_s^k) = [b_{i,s}^{(k)}, c_{i,s}^{(k)}),$$

$i = 0, \dots, h_k - 1,\ s = 1, \dots, r(k)$.

3.1 Construction of β. Let $\overline{\delta_k}$, $k \geq 1$, be positive numbers such that

$$\frac{\overline{\delta_k}}{|J_1^k|} < \min(1/2, \varepsilon_k/4),\ k \geq 1, \tag{20}$$

where $|J_1^k|$ denotes the length of J_1^k. We will inductively choose positive integers k_l, positive numbers δ_l and integers w_l, $l \geq 1$, so that

$$\delta_l \leq \overline{\delta}_{k_l},\ 0 \leq w_l \leq h_{k_l} - 1, \quad \max(1/h_{k_l}, w_l/h_{k_l}) < \varepsilon_{k_l}/4 \\ k_l < k_{l+1},\ l = 1, 2, \dots, \tag{21}$$

and

$$\begin{cases} \text{the closed intervals } B_l = [v_l - \delta_l/2, v_l + \delta_l/2], \\ v_l = c_{w_l, 8q(k_l)}^{(k_l)} \text{ form a decreasing sequence.} \end{cases} \tag{22}$$

Let us start with $k_1 = 1,\ w_1 = 0,\ \delta_1 = \overline{\delta}_1$. Suppose that we have defined $k_1, \dots, k_l,\ w_1, \dots, w_l$ and $\delta_1, \dots, \delta_l$ satisfying (21) and (22). Since $\tilde{T}$ is strictly ergodic, there exists a positive integer n such that for every $x \in [0, 1)$

$$\{x, \tilde{T}x, \dots, \tilde{T}^{n-1}x\} \cap \mathrm{Int}B_l \neq \emptyset. \tag{23}$$

We choose k_{l+1}, so that $k_{l+1} > k_l$ and $n/h_k < \varepsilon_k/4$, where $k = k_{l+1}$, which is possible in view of (16). Now, take $x = c_{0,8q(k_{l+1})}^{(k_{l+1})}$ and according to (23) choose $v_{l+1} = \tilde{T}^{w_{l+1}}(x) \in \mathrm{Int}(B_l)$. Then, we choose a positive number δ_{l+1} satisfying $\delta_{l+1} \leq \overline{\delta}_{k_l}$ and $B_{l+1} = [v_{l+1} - \delta_{l+1}/2, v_{l+1} + \delta_{l+1}/2] \subset \mathrm{Int}(B_l)$. Therefore, the sequences $\{\delta_l\}, \{w_l\}, \{k_l\}$ satisfy (21) and (22). Since $|B_l| \leq \overline{\delta}_{k_l} \longrightarrow 0$, there exists a unique $\beta \in [0,1)$ such that

$$\beta \in \bigcap_{l=1}^{\infty} B_l. \tag{24}$$

3.2 CONSTRUCTION OF $\tilde{\varphi}$ SATISFYING (9). We will define a sequence $\{\tilde{\varphi}_l\}_{l\geq 1}$ of cocycles, $\tilde{\varphi}_l$: $[0,1) \longrightarrow \mathbf{R}$ having disjoint supports. Let $a_1^{(k_l)}, \ldots, a_{r(k_l)}^{(k_l)}$ (the sequence $\{r(k_l)\}$ is determined by (3.1)) be real numbers satisfying

$$a_i^{(k_l)} = 0, \;\; i = 1, \ldots, q(k_l), \tag{25}$$

$$\sum_{s=1}^{8q(k_l)} a_s^{(k_l)} = 0, \tag{26}$$

$$a_{r\cdot 8q(k_l)+s}^{(k_l)} = 4^r \cdot a_s^{(k_l)}, \; s = 1, \ldots, 8q(k_l), \; r = 1, \ldots, p(k_l) - 1. \tag{27}$$

Let us define $\tilde{\varphi}_l$: $[0,1) \longrightarrow \mathbf{R}$ by

$$\begin{cases} \tilde{\varphi}_l(x) = 0 \;\; \text{if} \;\; x \in [0,1) \setminus I_{k_l}, \\ \tilde{\varphi}_l(x) = a_s^{(k_l)} \;\; \text{if} \;\; x \in J_s^{k_l}, \; s = 1, \ldots, r(k_l). \end{cases} \tag{28}$$

The conditions (18) and (25) guarantee that $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots$ have disjoint supports and the cocycle

$$\tilde{\varphi}(x) = \sum_{l=1}^{\infty} \tilde{\varphi}_l(x), \quad x \in [0,1) \tag{29}$$

is well defined.

THEOREM 2: *If the numbers $a_s^{(k_l)}$, $s = 1, \ldots, r(k_l)$, $l \geq 1$, satisfy (25), (26), (27) and $\tilde{\varphi}$ is defined by (28) and (29) then there exists a measurable function $\tilde{f} : \mathbf{R} \longrightarrow \mathbf{R}$ periodic of period 1 such that*

$$\tilde{f}(x + \alpha) - \tilde{f}(x) = \tilde{\varphi}(x + \beta) - 4\tilde{\varphi}(x), \quad x \in \mathbf{R}, \tag{30}$$

where β is defined by (24).

Proof: Put

$$\tilde{\psi}(x) = \tilde{\varphi}(x + \beta) - 4\tilde{\varphi}(x), \; \tilde{\psi}_l(x) = \tilde{\varphi}_l(x + \beta) - 4\tilde{\varphi}(x), \; x \in \mathbf{R}.$$

Then, we have $\sum_{l=1}^{\infty} \tilde{\psi}_l(x) = \tilde{\psi}(x)$.

In order to prove that $\tilde{\psi}$ is a coboundary it is enough to show that the assumptions of Corollary 2 are satisfied. Denote

$$Z_1 = \bigcup_{i=h_{k_l}-w_l}^{h_{k_l}-1} \tilde{T}(I_{k_l}),\ Z_2 = \bigcup_{i=0}^{h_{k_l}-1} \bigcup_{s=1}^{r(k_l)} [b_{i,s}^{(k_l)} - \delta_l/2, b_{i,s}^{(k_l)} + \delta_l/2],$$

$$Z_3 = \bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i\Big(\bigcup_{s=r(k_l)-8q(k_l)}^{r(k_l)} J_s^{k_l}\Big),\ Z_4 = I_{k_l}$$

and put

$$S_l = \bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i(I_{k_l}) \setminus (Z_1 \cup Z_2 \cup Z_3 \cup Z_4).$$

It follows from (21) that

$$\mu(Z_1) \le w_l/h_{k_l} \le \varepsilon_{k_l}/4.$$

In view of (20) and (21),

$$\mu(Z_2) \le \delta_l/|J_1^{k_l}| \le \varepsilon_{k_l}/4.$$

By (19) and (21)

$$\mu(Z_3) \le 1/p_{k_l} < \varepsilon_{k_l}/4,\ \mu(Z_4) < \varepsilon_{k_l}/4.$$

Therefore, by (14) and (16)

$$\mu(S_l) \ge 1 - \varepsilon_{k_l} - 4\frac{\varepsilon_{k_l}}{4} > 1 - \frac{1}{8}\cdot\frac{1}{2^{k_l}} \ge 1 - \frac{1}{2}\cdot\frac{1}{2^l}.$$

It remains to prove that

$$\text{(31)} \qquad \text{if } x, \tilde{T}^r x \in S_l \text{ then } \tilde{\psi}_l^{(r)} = 0,\ r \ge 1.$$

LEMMA 1: *Let $A = \tilde{T}^t J_p^{k_l}$ for some $0 \le t \le h_{k_l} - 1,\ 1 \le p \le r(k_l)$. Suppose that $x, \tilde{T}^m x \in A$ for some $m \ge 1$. Then*

$$\text{(32)} \qquad \tilde{\varphi}_l^{(m)} = 0.$$

Proof: We will divide the trajectory $\{x, \tilde{T}x, \dots, \tilde{T}^{m-1}x\}$ of x into a disjoint union of subsets according to the return times of x into A. More precisely,

$$\{x, \tilde{T}x, \dots, \tilde{T}^{m-1}x\} = \{\tilde{T}^{n_0}x, \dots, \tilde{T}^{n_1-1}x\} \cup \{\tilde{T}^{n_1}x, \dots, \tilde{T}^{n_2-1}x\} \cup \dots \cup \{\tilde{T}^{n_{u-1}}x, \dots, \tilde{T}^{n_u-1}x\},$$

where $\tilde{T}^{n_i}x$ ($i = 0, \ldots, u-1$) is the only point of $\{\tilde{T}^{n_i}x, \ldots, \tilde{T}^{n_{i+1}-1}x\}$ belonging to A, $n_0 = 0$, $n_u = m$. Notice that if $y \notin \bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i I_{k_l}$ and if p is the first return time of y to $\bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i I_{k_l}$, then necessarily $y \in J_1^{k_l}$ since the latter set is the base of a Rokhlin tower refining the set $\bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i I_{k_l}$. Since $\tilde{\varphi}_l$ vanishes outside of I_{k_l}, for $i = 0, \ldots, u-1$,

$$\tilde{\varphi}_l(\tilde{T}^{n_i}x) + \tilde{\varphi}_l(\tilde{T}^{n_i+1}x) + \ldots + \tilde{\varphi}_l(\tilde{T}^{n_{i+1}-1}x) = \sum_{j=1}^{r(k_l)} a_j^{(k_l)}.$$

The latter sum, in view of (25), (26) and (27) is equal to 0. Consequently (32) holds true. ∎

LEMMA 2: *Let $A = \tilde{T}^t J_p^{k_l} \cap S_l$ $(0 \le t \le h_{k_l} - w_l - 1,\ 1 \le p \le r(k_l) - 1)$ and suppose that $x, \tilde{T}^m x \in A$ for some $m \ge 1$. Then*

$$\tilde{\varphi}_l^{(m)}(x) = \tilde{\varphi}_l^{(m)}(\tilde{S}x) = 0, \text{ where } \tilde{S}x = x + \beta.$$

Proof: Notice that if $x, \tilde{T}^m x \in A$ then $\tilde{S}x, \tilde{T}^m(\tilde{S}x) \in \tilde{T}^{t+w_l}(J_{p+1}^{k_l})$ because $\tilde{S}$ commutes with $\tilde{T}$. Therefore, the assertion follows from Lemma 1. ∎

Assume that $x \in A = \tilde{T}^t J_p^{k_l} \cap S_l$, $\tilde{T}^r x \in B = \tilde{T}^u J_q^{k_l} \cap S_l$ for some $0 \le t, u \le h_{k_l} - w_l - 1$, $1 \le p, q \le r(k_l) - 1$. Two different cases arise.

CASE 1: $q \le p$ or $u \le t$ if $p = q$. Let $n \ge 0$ be the smallest positive integer such that $\tilde{T}^{r+n}x \in \tilde{T}^t J_p^{k_l}$. Notice that then $\tilde{T}^{r+n}x \in A$ since $\tilde{T}$ is an isometry. We have

$$\tilde{\psi}_l^{(r+n)}(x) = \tilde{\psi}_l^{(r)}(x) + \tilde{\psi}_l^{(n)}(\tilde{T}^r x).$$

In view of Lemma 2, $\tilde{\psi}_l^{(r+n)}(x) = 0$ and hence it is enough to prove that $\tilde{\psi}_l^{(n)}(x) = 0$, where $y = \tilde{T}^r x$, and $y, \tilde{T}^n y \in S_l$.

CASE 2: $p \le q$ or $t \le u$ if $p = q$. Let $n \ge 0$ be the smallest positive integer such that $\tilde{T}^n x \in \tilde{T}^u J_q^{k_l}$. Then, in fact, $\tilde{T}^n x \in B$. We have

$$\tilde{\psi}_l^{(r)}(x) = \tilde{\psi}_l^{(n)}(x) + \tilde{\psi}_l^{(r-n)}(\tilde{T}^n x).$$

Since, $\tilde{T}^n x \in B$, $\tilde{T}^{(r-n)+n}x \in B$, by Lemma 2, $\tilde{\psi}_l^{(r-n)}(y) = 0$, where $y = x$, and $y, \tilde{T}^n y \in S_l$.

Therefore, we have to prove that if $y, T^n y \in S_l$ with $n = 1, \ldots, N-1$, where N is the smallest positive integer such that $\tilde{T}^N y \notin \bigcup_{i=0}^{h_{k_l}-1} \tilde{T}^i I_{k_l}$ then

$$\tilde{\psi}_l^{(n)}(y) = 0. \tag{33}$$

It follows from (24) and (28) that $\tilde{\psi}_l$ vanishes on S_l except for $I_{k_l} \cap S_l$ and $\tilde{T}^{h_{k_l}-w_l} I_{k_l} \cap S_l$ and moreover

$$\tilde{\psi} \mid J_p^{k_l} \cap S_l = -4a_p^{(k_l)}, \quad \tilde{\psi} \mid \tilde{T}^{h_{k_l}-w_l} J_p^{k_l} \cap S_l = a_{q(k_l)+p+1}^{(k_l)} = 4a_{p+1}^{(k_l)}.$$

In view of definition of S_l and n, it follows that (33) holds and the proof of Theorem 2 is complete. ∎

3.3 Construction of φ satisfying (8) and (9). According to Theorem 2, for an arbitrary choice $\{a_1^{(k_l)}, \ldots, a_{r(k_l)}^{(k_l)}\}$ $(l \geq 1)$ satisfying its assumptions we obtain $\tilde{\varphi}$ satisfying (9). In this section, we put some restrictions on these parameters to get (8).

For $Z_5 = \{0,1,2,3,4\}$, let $v\colon Z_5 \longrightarrow Z_5$ be a group automorphism given by

$$v(i) = 4i \ (\mathrm{mod}\ 5). \tag{34}$$

Partition the set of natural numbers in any way into

$$\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2, \quad \text{where } \mathbf{N}_1, \mathbf{N}_2 \text{ are infinite.} \tag{35}$$

For $l \in \mathbf{N}_1$ we put

$$\begin{cases} a_{q(k_l)}^{(k_l)} = a_{2q(k_l)+1}^{(k_l)} = \frac{1}{5} \\ \text{and for all other } i = 1, \ldots, 7q(k_l) - 1 \\ a_i^{(k_l)} = 0, \\ \text{for } t = 7q(k_l), \ldots, 8q(k_l) \text{ put} \\ a_t^{(k_l)} = \frac{m_t}{5},\ m_t \text{ arbitrary, } m_t \in \mathbf{Z} \text{ with } \sum_{j=1}^{8q(k_l)} a_j^{(k_l)} = 0 \end{cases} \tag{36}$$

((27) forces the rest of the $a_j^{(k_l)}$).

Theorem 3: *If φ satisfies the assumptions of Theorem 2 and, for $l \in \mathbf{N}_1$, the cocycle $\tilde{\varphi}$ is defined according to (36) then φ satisfies (8) and (9).*

Proof: We start with the following.

Lemma 3: *If the numbers $a_s^{(k_l)}$, $1 \leq s \leq r(k_l)$, $l \geq 1$, satisfy (25), (26) and (27) then the cocycle $\tilde{\varphi}$ defined by (28) and (29) is constant on each interval $\tilde{T}^i(I_{k_l})$, $i = 1, \ldots, h_{k_l} - 1$. Moreover, if we put $b_i^{(k_l)} = \tilde{\varphi} \mid \tilde{T}^i(I_{k_l})$ then*

$$\sum_{i=1}^{h_{k_l}-1} b_i^{(k_l)} = 0 \tag{37}$$

holds.

Proof: First, we will prove that

(38) for each $p=0,\ldots,h_{k_l}-1$ either $\tilde{T}^p(I_{k_l})\subset X\setminus\bigcup_{i=0}^{h_{k_{l-1}}}\tilde{T}^i I_{k_{l-1}-1}$ or there exists i, $0\le i\le h_{k_{l-1}}\cdot r(k_{l-1})-1$ such that $\tilde{T}^p(I_{k_l})\subset\tilde{T}^i(J_1^{k_{l-1}})$.

Indeed, notice that (38) holds for $p=0,\ldots,h_{k_{l-1}}\cdot r(k_{l-1})-1$ since $I_{k_l}\subset J_1^{k_{l-1}}$. Denote $B_l=\tilde{T}^{h_{k_{l-1}}\cdot r(k_{l-1})-1}(I_{k_l})$. For each $x\in B_l$ let $n(x)$ be the first return time of x into $\bigcup_{i=0}^{h_{k_{l-1}}-1}\tilde{T}^i I_{k_{l-1}}$. Consequently, $\tilde{T}^{n(x)}x\in J_1^{k_{l-1}}$. Put $n=\min_{x\in B_l}n(x)$. Thus, $\tilde{T}B_l,\ldots,\tilde{T}^{n-1}B_l$ are contained in $X\setminus\bigcup_{i=0}^{h_{k_{l-1}}-1}\tilde{T}^i I_{k_{l-1}}$ and $\tilde{T}^n B_l\cap J_1^{k_{l-1}}\neq\emptyset$. If $\tilde{T}^n B_l$ is still a member of the tower $\xi_{k_l}=(I_{k_l},\ldots,\tilde{T}^{h_{k_l}-1}I_{k_l})$ then $I_{k_l}\cap\tilde{T}^n B_l=\emptyset$ and since $\tilde{T}^n B_l\cap J_2^{k_{l-1}}=\emptyset$ and $\tilde{T}$ is an isometry, it follows that $\tilde{T}^n B_l\subset J_1^{k_{l-1}}$. We can keep this argument going as long as we deal with the intervals of ξ_{k_l}. Hence (38) has been proved.

Now, notice that whenever $\tilde{T}^s(I_{k_l})\subset J_1^{k_{l-1}}$, the intervals

$$\tilde{T}^{s+1}(I_{k_l}),\ldots,\tilde{T}^{s+h_{k_{l-1}}\cdot r(k_{l-1})-1}(I_{k_l})$$

are members of ξ_{k_l} (see (18)). This observation shows that the values of $\tilde{\varphi}_{k_{l-1}}$ on

$$\tilde{T}I_{k_l},\ldots,\tilde{T}^{h_{k_l}-1}I_{k_l}$$

are constant with sum equal to zero, since $\sum_{i=1}^{r(k_{l-1})}a_i^{(k_{l-1})}=0$.

We can repeat the same argument for $l-2,l-3,\ldots,1$, which completes the proof.

Suppose now, that for some $V\in C(T)$ there exists a measurable $g\colon[0,1)\longrightarrow X$ satisfying (8) and $Vx=x\cdot e^{2\pi i\gamma},\gamma\in[0,1)$. Denote

$$\eta_{k_l}=\{J_1^{k_l},\tilde{T}J_1^{k_l},\ldots,\tilde{T}^{h_{k_l}\cdot r(k_l)-1}J_1^{k_l}\}.$$

Given $\varepsilon>0$, there exists l_0 such that for all $l\ge l_0$, $l\in\mathbf{N}_1$ there exist at least $(1-\varepsilon)h_{k_l}\cdot r(k_l)$ intervals of η_{k_l} on which the values of function g are contained in a ball of radius ε except for $\frac{\varepsilon}{|J_1^{k_l}|}$ of the mass of the interval. (This is a consequence of the measurability of g). Therefore, it is possible to find an interval $A=\tilde{T}^i(J_{s_0}^{k_l})$ for some $1\le s_0\le\frac{2}{3}r_{k_l}$, $1\le i\le h_{k_l}-1$ with the following properties

(39) $$\begin{cases}|g(x)-c_1|<\varepsilon\text{ for }x\in A\text{ except for a subset}\\ \text{of }A\text{ of measure }<\varepsilon\cdot\mu(A),\end{cases}$$

$$(40)\quad \tilde{V}(A) \subset \bigcup_{i=0}^{h_{k_l}-1} \bigcup_{s=0}^{[\frac{2}{3}r(k_l)]} \tilde{T}^i(J_s^{k_l} \quad) \cap \tilde{T}^j(I_{k_l}) \quad \text{for some } 1 \leq j \leq h_{k_l}-1,$$

and

$$(41)\quad \begin{cases} |g(x)-c_1| < \varepsilon \; for \; x \in \tilde{T}^i(I_{k_l}) \text{ and } |g(x)-c_2| < \varepsilon \text{ for } x \in \tilde{T}^j(I_{k_l}) \\ \text{except for sets of measure } < \varepsilon \cdot \mu(I_{k_l}), \end{cases}$$

where l is large enough, $l \in \mathrm{N}_1$. Let $B \subset \tilde{T}^j(I_{k_l})$ be that interval of η_{k_l} for which

$$(42)\qquad \mu(\tilde{V}(A) \cap B) > \frac{1}{2}\mu(B).$$

Let us consider first the possible case $\varphi(x)/\varphi_1(\tilde{V}x) = g(\tilde{T}x)/g(x)$, where $\varphi_1 = \varphi^2$. We have

$$(43)\qquad \frac{\varphi^{(s\cdot h_{k_l})}(x)}{\varphi_1^{(s\cdot h_{k_l})}(\tilde{V}x)} = \frac{g(\tilde{T}^{s\cdot h_{k_l}}x)}{g(x)}$$

for every $x \in A \cap \tilde{V}^{-1}(B)$ and every $s = 1, \ldots, [\frac{1}{3}r(k_l)]$.

Consider the set Λ of those $s = 1, \ldots, [\frac{1}{3}r(k_l)]$ for which $|g(x) - c_1| < \varepsilon$ for $x \in \tilde{T}^i(J_s^{k_l})$ except for a set of measure $< \varepsilon \cdot \mu(J_1^{k_l})$. We then have

$$(44)\qquad \operatorname{card}(\Lambda) > (1-3\varepsilon)\frac{r(k_l)}{3}$$

For any $s \in \Lambda$ we can make the following computations. Since (42) holds, we can find an $x \in A \cap \tilde{V}^{-1}(B)$ for which (43) holds and $|g(x)-c_1| < \varepsilon$, $|g(\tilde{T}^{s\cdot h_{k_l}}x)-c_1| < \varepsilon$. Consequently

$$\left|\frac{g(\tilde{T}^{s\cdot h_{k_l}}x)}{g(x)} - 1\right| < 2\varepsilon$$

and by (43)

$$(45)\qquad \left|\frac{\varphi^{(s\cdot h_{k_l})}(x)}{\varphi_1^{(s\cdot h_{k_l})}(\tilde{V}x)} - 1\right| < 2\varepsilon\ .$$

(actually, the set of x satisfying (45) is of positive measure).

Now, we will calculate $\varphi^{(sh_{k_l})}(x)$ and $\varphi_1^{(sh_{k_l})}(\tilde{V}x)$. We have,

$$\tilde{\varphi}(y) = \tilde{\varphi}_l(y) = a_w^{(k_l)} \text{ if } y \in J_w^{k_l} \text{ and } 1 \leq w \leq r(k_l).$$

Hence, by Lemma 3, for an appropriate $u \leq \frac{2}{3}r(k_l)$ we obtain that

$$\varphi^{(sh_{k_l})}(x) = \exp[2\pi i(a_{s_0+1}^{(k_l)} + \ldots + a_{s_0+s}^{(k_l)})],$$

$$\varphi_1^{(sh_{k_l})}(\tilde{V}x) = \exp[2\pi i \cdot 2(a_u^{(k_l)} + \cdots + a_{u+s-1}^{(k_l)})].$$

Denote

$$d_j^{(l)} = e^{2\pi i \cdot a_j^{(k_l)}}, \quad L_s = \frac{d_{s_0+1}^{(l)} \cdot \ldots \cdot d_{s_0+s}^{(l)}}{(d_u^{(l)} \cdot \ldots \cdot d_{u+s-1}^{(l)})^2}.$$

In view of (45), we have $|L_s - 1| < 2\varepsilon$ whenever $s \in \Lambda$, so

$$L_s = 1 \quad \text{for } s \in \Lambda \tag{46}$$

(all the numbers L_s are fifth roots of unity). Since (44) holds, there exists $1 \le s \le [\frac{1}{3}r(k_l)] - 8q(k_l)$ such that $s + s_0 = m \cdot 8q(k_l)$ and

$$\operatorname{card}(\{s+1, \ldots, s+8q(k_l)\} \cap \Lambda) \ge (1 - 4\varepsilon)8q(k_l). \tag{47}$$

Let $s_1 = s_0 + s + q(k_l)$, $s_2 = s_0 + s + 2q(k_l) + 1$. Therefore, for an appropriate $\omega_1 \in \mathbf{N}$

$$a_{s_1}^{(k_l)} = a_{s_2}^{(k_l)} = 4^{\omega_1} \cdot \frac{1}{5} \quad \text{and for all } j,\ |j - s_i| < q(k_l)\ (i = 1,2),\ a_j^{(k_l)} = 0.$$

Consequently, there exists $i \in \{0,1\}$ such that for all j, $|j - s_i| < \frac{1}{2}q(k_l)$ the values $a_{u+s+j}^{(k_l)}$ are all equal to zero except for at most one value of the form $4^{\omega_2} \cdot \frac{1}{5}$. But

$$L_p = L_{p-1} \cdot \frac{d_{s_0+p}^{(l)}}{(d_{u+p-1}^{(l)})^2}$$

and since

$$\{4^t (\text{mod} 5) \colon t \ge 0\} = \{1, 4\}, \ \{2 \cdot 4^t (\text{mod} 5) \colon t \ge 0\} = \{2, 3\},$$

$\operatorname{card}\{1 \le j \le 2q(k_l) \colon s + j \notin \Lambda\} \ge \frac{1}{2}q(k_l)$, which contradicts (47).

To complete the proof, instead of (43) we must also consider the equation

$$\varphi^{(sh_{k_l})}(x) \cdot \varphi_1^{(sh_{k_l})}(\tilde{V}x) = g(\tilde{T}^{sh_{k_l}}x)/g(x). \tag{48}$$

We obtain a contradiction in exactly the same way as for (43), because $\{-2 \cdot 4^t (\text{mod} 5) \colon t \ge 0\} = \{2, 3\}$. ∎

3.4 Construction of φ also satisfying (6). We still have some freedom in our construction since for $l \in \mathbf{N}_2$ the values $(a_1^{(k_l)}, \ldots, a_{r(k_l)}^{(k_l)})$ are arbitrary subject to satisfying the assumptions from (3.2). Partition the set $\mathbf{N}_2$ into

$$\mathbf{N}_2 = \bigcup_{t=1}^{\infty} \mathbf{N}_{2,t} \text{ and each of } \mathbf{N}_{2,t} \text{ is infinite.} \tag{49}$$

Let $\mathcal{P} = \{p_t \colon t \ge 1\}$ be an infinite set of prime numbers. For $l \in \mathbf{N}_{2,t}$ we define $(a_1^{(k_l)}, \ldots, a_{8q(k_l)}^{(k_l)})$ by (36), where 5 is replaced by p_t, $t \ge 1$.

THEOREM 4: *If φ satisfies the assumptions of Theorem 3 and is defined for $l \in \mathbf{N}_2$ as above then φ satisfies (6), (8) and (9).*

Proof: We have to prove (6), i.e. the ergodicity of T_φ. Suppose that T_φ is not ergodic. In view of (5), there exists an integer $n \neq 0$ and a measurable $g: X \longrightarrow X$ satisfying

$$(\varphi(x))^n = g(Tx)/g(x).$$

Take $p_t \in \mathcal{P}$ such that p_t does not divide n and consider only $l \in \mathbf{N}_{2,t}$. By repeating the arguments from the proof of Theorem 3, given $\varepsilon > 0$, for l large enough there exists $s_0 \leq \frac{2}{3}r(k_l)$ such that

$$|(d_{s_0}^{(k_l)} \cdot \ldots \cdot d_{s_0+s}^{(k_l)})^n - 1| < 2\varepsilon,$$

where $s \in \Lambda \subset \{1, \ldots, [\frac{1}{3}r(k_l)]\}$ and

$$\operatorname{card}(\Lambda) \geq (1 - 3\varepsilon)r(k_l)/3.$$

Denote $L_s = (d_{s_0}^{(k_l)}, \ldots, d_{s_0+s}^{(k_l)})^n$ and observe that $L_s = L_{s-1} \cdot (d_{s_0+s}^{(k_l)})^n$. Since $s \in \Lambda$ iff $L_s = 1$ (because $\gcd(p_t, n) = 1$ and L_s are p_t-roots of unity), we can obtain a contradiction proceeding as at the end of the proof of Theorem 3. ∎

4. A class of R-cocycles which can be smoothed

Assume that for given sequences $\{\varepsilon_k\}$ satisfying (14) and $\{C_k\}$ to be specified later, we have an $\alpha \in [0, 1)$ satisfying the (R) condition. We will consider cocycles $\tilde{\varphi}$ given as

$$\tilde{\varphi}(x) = \sum_{k=1}^{\infty} \tilde{\varphi}_k(x),$$

where $\tilde{\varphi}_k$ is zero outside of I_k and $\tilde{\varphi}_k | J_t^k = a_t^{(k)}$, $t = 1, \ldots, r(k)$, i.e. $\tilde{\varphi}_k$ is completely determined by $(a_1^{(k)}, \ldots, a_{r(k)}^{(k)}) \in \mathbf{R}^{r(k)}$. We assume that

$$\sum_{t=1}^{r(k)} a_t^{(k)} = 0 \tag{50}$$

(*i.e.* each $\tilde{\varphi}_t$ is an $\mathbf{R}$-coboundary) . We will also assume that $\tilde{\varphi}$ is bounded, so there exists $K > 0$ such that

$$|a_t^{(k)}| \leq K, \quad k \geq 0, \quad t = 1, \ldots, r(k). \tag{51}$$

Remark 5: Note that the cocycles from the proof of Theorem 4 are unbounded. However, by taking

$$\tilde{\tilde{\varphi}}_l(x):=\text{frac}(\tilde{\varphi}_l(x)),\quad x\in\mathbf{R},$$

where frac(c) denotes the fractional part of $c\geq 0$ and frac(c) $=-$frac($-c$) if $c<0$ we obtain an $\mathbf{R}$-cocycle which is certainly bounded but the sum in (50) is not zero for it. However this sum is an integer. Therefore, by either adding $+1$ to appropriately chosen negative components or subtracting $+1$ from appropriately chosen positive components we get a bounded by 1 cocycle $\tilde{\psi}$ for which (50) holds and $\exp(2\pi i\tilde{\varphi}(x))=\exp(2\pi i\tilde{\psi}(x))$. Therefore, for the cocycle $\psi=\exp(2\pi i\tilde{\psi})$ we have (6) and (8) although probably (9) is no longer true for $\tilde{\psi}$. However, we need rather (7) than (9) to hold. The fact that statement (6) holds true for ψ follows from the following assertion whose proof is very like to the proof of Proposition 3. ∎

Fact: Suppose that we are given a sequence of cocycles $\tilde{\varphi}_k\colon X\longrightarrow\mathbf{R}$, $k=1,2,\ldots$ such that the cocycle $\tilde{\varphi}=\sum_{k=1}^{\infty}\tilde{\varphi}_k$ is well defined. Assume that there exist measurable sets S_k, $\mu(S_k)>1-\delta_k$, $\sum_{k=1}^{\infty}\delta_k<1$ such that if $x, T^tx\in S_k$ then $\varphi_k^{(t)}(x)=1$, where $\varphi_k=\exp(2\pi i\tilde{\varphi}_k)$. Then the cocycle

$$\varphi(x)=\exp(\sum_{k=1}^{\infty}\tilde{\varphi}_k(x))$$

is an $\mathbf{S}^1$-coboundary.

Note also that the cocycles from the proof of Theorem 4 are defined on a subsequence of the towers ξ_k. This is the same as saying that $\tilde{\varphi}_k=0$ for $k\notin\{k_l\colon l\in\mathrm{N}\}$. ∎

We will put some restrictions on the sequences $\{r(k)\}$ and $\{a_1^{(k)},\ldots,a_{r(k)}^{(k)}\}$ to prove that in the cohomology class of $\tilde{\varphi}$ there is a C^{∞}-cocycle.

Let $\{M_k\}$ be a sequence of integers, $M_k\geq 2$ such that only M_k of the numbers $a_1^{(k)},\ldots,a_{r(k)}^{(k)}$ are different from zero. Let R_k be integers such that

$$R_k>4\cdot 2^k,\quad k\geq 1. \tag{52}$$

Let $\delta_k=\frac{1}{4\cdot 2^k}$ and

$$\eta_k=\delta_k/M_k. \tag{53}$$

Let $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ be a periodic function of period 1 of C^∞-class satisfying

$$\frac{d^m f}{dx^m}(0) = 0\,, \quad f \geq 0\,, \quad c = \int_0^1 f(x)dx > 0, \tag{54}$$

$m \geq 0$.

Remark 6: Such a function cannot be analytic. An example of a function satisfying (54) is $f(x) = \exp(-(\sin 2\pi x)^{-2})$. ∎

Put

$$B_m = \sup_{x\in[0,1)} \left|\frac{d^m f(x)}{dx^m}\right|, \quad m \geq 0. \tag{55}$$

Take $n \geq 2$ and denote

$$\Delta = \Delta_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\},$$

$$\sigma(f, \Delta, \{x_i\}) = \frac{1}{n}\sum_{i=0}^{n-1} f(x_i),$$

where $x_i \in [\frac{i}{n}, \frac{i+1}{n})$, $i = 0, \ldots, n-1$. Since f is continuous,

$$\begin{cases} \text{given } \varepsilon > 0 \text{ there is } n_0 \text{ such that for } n \geq n_0 \\ |\sigma(f, \Delta_n, \{x_i\}) - c| < \varepsilon \text{ for every } \{x_i\}. \end{cases} \tag{56}$$

Apply (56) with $\varepsilon = \eta_k$ to get a sequence of integers n'_k such that

$$|\sigma(f, \Delta_n, \{x_i\}) - c| < \eta_k \ \text{ for } n \geq n'_k,\ k \geq 1. \tag{57}$$

Let α satisfy the (R) condition with respect to $\{\varepsilon_k\}$ and

$$C_k = \frac{(M_k R_k)^{k+1}}{(1-\varepsilon_k)^k}. \tag{58}$$

Actually, no harm arises if we assume that additionally

$$r(k) = n_k M_k R_k, \tag{59}$$

where

$$n_k \geq n'_k, \quad k \geq 1. \tag{60}$$

Suppose that some pairwise disjoint subintervals $w_1^{(k)}, w_2^{(k)}, \ldots, w_{M_k}^{(k)}$ of I_k, $k \geq 1$ are given and the following properties

$$w_i^{(k)} = J_{s_i}^k \cup \ldots \cup J_{s_i+n_k-1}^k \quad \text{for some } s_i \geq 2,\ s_i + n_k < s_{i+1}. \tag{61}$$

$$(\exists!\ s_i \le \overline{s_i} \le s_i + n_k - 1) \quad a^{(k)}_{\overline{s_i}} \ne 0 \tag{62}$$

(*i.e.* $\tilde{\varphi}_k$ takes a nonzero value on only one subinterval in each $w_i^{(k)}$) hold. Put

$$c_t^{(k)} = a_t^{(k)}/(c \cdot n_k), \quad t = 1, \dots, r(k)\ . \tag{63}$$

Consequently, by (50)

$$\sum_{t=1}^{r(k)} c_t^{(k)} = 0\ . \tag{64}$$

Let us define a C^∞-function $g_{i,k} = g_{i,k}(x)$, $i = 1, \dots, M_k$ as follows

$$g_{i,k}(x) = c^{(k)}_{\overline{s_i}} \cdot f\left(\frac{x - a_{i,k}}{b_{i,k} - a_{i,k}}\right) \text{ if } x \in [a_{i,k}, b_{i,k}) = w_i^{(k)} \text{ and } 0 \text{ otherwise}$$

and then put

$$\tilde{\psi}_k(x) = g_{i,k}(x) \text{ if } x \in w_1^{(k)} \cup \cdots \cup w_{M_k}^{(k)} \text{ and } 0 \text{ otherwise.} \tag{65}$$

The functions $\tilde{\psi}_k$ are of C^∞-class and they have disjoint supports. Therefore

$$\tilde{\psi}(x) = \sum_{k \ge 1} \tilde{\psi}_k(x)\ , \quad x \in [0, 1)$$

is well-defined and $\tilde{\psi}(0) = \tilde{\psi}(1) = 0$. Note that

$$\frac{d^m \tilde{\psi}_k(x)}{dx^m} = c^{(k)}_{\overline{s_i}} \cdot \frac{d^m f\left(\frac{x - a_{i,k}}{b_{i,k} - a_{i,k}}\right)}{dx^m} \cdot \frac{1}{|w_i^{(k)}|^m}\ , \quad \text{if } x \in w_i^{(k)}.$$

Therefore, by (55), (63) and (51), (61) we have

$$\left|\frac{d^m \tilde{\psi}_k(x)}{dx^m}\right| \le |c^{(k)}_{\overline{s_i}}| \cdot B_m \cdot \frac{1}{|w_i^{(k)}|^m}$$

$$= \frac{|a^{(k)}_{\overline{s_i}}|}{c \cdot n_k} \cdot B_m \cdot \frac{1}{|w_i^{(k)}|^m} \le \frac{K}{c \cdot n_k} \cdot B_m \cdot \frac{1}{(n_k \cdot |J_1^k|)^m}.$$

But , in view of (59)

$$|J_1^k| \ge \frac{1 - \varepsilon_k}{h_k \cdot r(k)} = \frac{1 - \varepsilon_k}{h_k \cdot n_k \cdot M_k \cdot R_k}$$

so

$$\left|\frac{d^m \tilde{\psi}_k(x)}{dx^m}\right| \le \frac{K}{cn_k} \cdot B_m \left(\frac{M_k \cdot R_k \cdot h_k}{1 - \varepsilon_k}\right)^m \le \frac{K}{c \cdot n_k} \cdot B_m \left(\frac{M_k \cdot R_k \cdot h_k}{1 - \varepsilon_k}\right)^k$$

for $k \geq m$. Hence, in view of (58) and (R1) the series $\sum_{k=1}^{\infty} \frac{d^m \tilde{\psi}_k(x)}{dx^m}$ is uniformly convergent and consequently $\tilde{\psi}$ is of C^∞-class.

THEOREM 5: *Let $\alpha \in [0,1)$ be irrational. Assume that α satisfies the* (R) *condition with respect to $\{\varepsilon_k\}$, $\{C_k\}$ and (14), (59), (52), (60), (58) hold. If a cocycle $\tilde{\varphi} = \sum_{k\geq 1} \tilde{\varphi}_k$ is given by the sequences $\{(a_1^{(k)}, \ldots, a_{r(k)}^{(k)})\}$ satisfying (50), (51), (61) and (62), then there exists $\tilde{\psi}$: $\mathbf{R} \to \mathbf{R}$ periodic of period 1, of C^∞-class such that*

$$\tilde{\psi}(x) - \tilde{\varphi}(x) = \tilde{h}(x+\alpha) - \tilde{h}(x)$$

for a 1−periodic measurable $\tilde{h} : \mathbf{R} \longrightarrow \mathbf{R}$.

Proof: Put $\tilde{\psi}(x) = \sum_{k\geq 1} \tilde{\psi}_k(x)$, $x \in [0,1)$, where $\tilde{\psi}_k$ is defined by (65). We have already shown that under our standing assumptions, $\tilde{\psi}(0) = \tilde{\psi}(1)$ and $\tilde{\psi}$ is of C^∞-class. It remains to prove that the $\mathbf{R}$-cocycle $\tilde{\varphi} - \tilde{\psi}$ is an $\mathbf{R}$-coboundary. We have

$$(\tilde{\psi} - \tilde{\varphi})(x) = \sum_{k\geq 1} (\tilde{\psi}_k(x) - \tilde{\varphi}_k(x)).$$

We will show that the cocycles $\tilde{\psi}_k - \tilde{\varphi}_k$, $k \geq 1$, satisfy the assumptions of Corollary 2. Put

$$S_k = \bigcup_{i=0}^{h_k - 1} \tilde{T}^i(I_k \setminus (w_1^{(k)} \cup \cdots \cup w_{M_k}^{(k)})).$$

Note that if $x \in J_1^k$ then

$$\sum_{i=0}^{r(k)\cdot h_k - 1} \tilde{\psi}_k(\tilde{T}^i x) = 0 \tag{66}$$

because

$$\sum_{i=0}^{r(k)\cdot h_k - 1} \tilde{\psi}_k(\tilde{T}^i x) = \sum_{i=0}^{M_k} \sum_{j=0}^{n_k - 1} \tilde{\psi}_k(\tilde{T}^{(s_i + j - 1)n_k}(x)) = \sum_{j=0}^{n_k - 1} (\sum_{i=1}^{M_k} c_{\overline{s}_i}^{(k)} \cdot f(t_j)),$$

where $c_{\overline{s}_i}^{(k)} \cdot f(t_j)$ is the common value of

$$g_{1,k}(\tilde{T}^{(s_1 + j - 1)h_k}(x)) = \cdots = g_{M_k,k}(\tilde{T}^{(s_{M_k} + j - 1)h_k}(x))$$

and (64) holds.

Now, take $x \in J^k_{s_i}$ and note that

$$|\tilde{\psi}_k^{(n_k \cdot h_k+1)}(x) - \tilde{\varphi}_k^{(n_k \cdot h_k+1)}(x)| < \frac{K}{c} \cdot \eta_k, \tag{67}$$

where $i = 1, \dots, M_k$. Indeed,

$$\tilde{\psi}_k^{(n_k \cdot h_k+1)}(x) = \sum_{j=0}^{n_k} \tilde{\psi}_k(\tilde{T}^{jh_k}(x) = \sum_{j=0}^{n_k} g_{i,k}(\tilde{T}^{jh_k}(x)) = c^{(k)}_{\bar{s}_i} \sum_{j=0}^{n_k} f(y^{(j)}),$$

where

$$y^{(j)} = \frac{\tilde{T}^{jh_k}(x) - a_{i,k}}{b_{i,k} - a_{i,k}}.$$

Now

$$\frac{j}{n_k} \le y^{(j)} \le \frac{j+1}{n_k}, \quad j = 0, \dots, n_k - 1$$

and (60), (56) imply that

$$|\frac{1}{n_k} \sum_{j=0}^{n_k-1} f(y^{(j)}) - c| < \eta_k. \tag{68}$$

Moreover, $\tilde{\varphi}_k^{(n_k \cdot h_k+1)}(x) = a^{(k)}_{\bar{s}_i}$ as soon as $x \in J^k_{s_i}$, $i = 1, \dots, M_k$. Hence, by (63), (51) and (68)

$$\begin{aligned}
|\tilde{\psi}_k^{(n_k \cdot h_k+1)}(x) - \tilde{\varphi}_k^{(n_k \cdot h_k+1)}(x)| &= |c^{(k)}_{\bar{s}_i} \cdot \sum_{j=0}^{n_k-1} f(y^{(j)}) - a^{(k)}_{\bar{s}_i}| \\
&= |a^{(k)}_{\bar{s}_i}| \cdot |\frac{1}{c \cdot n_k} \sum_{j=0}^{n_k-1} f(y^{(j)}) - 1| \le \frac{K}{c} \eta_k
\end{aligned}$$

and (67) holds.

Suppose, now, that $x, \tilde{T}^r x \in S_k$. Since $\tilde{\psi}_k, \tilde{\varphi}_k$ vanish outside of $w_1^{(k)} \cup \dots \cup w_{M_k}^{(k)}$, and (66), (50) and (67) hold,

$$|\tilde{\psi}_k^{(r)}(x) - \tilde{\varphi}_k^{(r)}(x)| \le 2M_k[\frac{K}{c} \cdot \eta_k],$$

because if a point z returns to $\bigcup_{i=0}^{h_k-1} \tilde{T}^i I_k$, necessarily, it falls into J^k_1. Since (53) holds, the theorem follows. ∎

COROLLARY 3: *There exist an irrational α and a cocycle $\varphi\colon \mathbf{R} \longrightarrow \mathbf{R}$ of C^∞-class such that*

(i) $T_{\exp(2\pi i\cdot\tilde{\varphi})}$ *is ergodic,* $Tx = x\cdot e^{2\pi i\alpha}$, $x \in X$,

(ii) $T_{\exp(2\pi i\cdot\tilde{\varphi})}$ *has a weakly isomorphic but not isomorphic factor.*

Proof: This is only to apply Theorem 5, Remark 5 and Theorem 4 to α satisfying the full (R) condition. ∎

5. A note on the smooth centralizer of smooth Anzai skew products

Assume that $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is a periodic function of period 1 and of C^∞-class such that

$$T_\varphi = T_{\exp(2\pi i\tilde{\varphi})}\colon (X\times X,\tilde{\mu}) \longrightarrow (X\times X,\tilde{\mu}) \text{ is ergodic.} \tag{69}$$

According to [2], [19] if $\tilde{S} \in C(T_\varphi)$ then there exist $S \in C(T)$, $f\colon X \longrightarrow X$ measurable and $n \in \mathbf{Z}\setminus\{0\}$ such that

$$\tilde{S}(x,y) = S_{f,n}(x,y) = (Sx, f(x)\cdot y^n). \tag{70}$$

We want to work with $\tilde{S} \in C(T_\varphi)$ which are smooth, equivalently, with $\tilde{S}$'s for which f is smooth. Such $\tilde{S}$'s are necessarily invertible. Indeed, if f is continuous, then $S_{f,n}$ is a continuous map commuting with T_φ and T_φ is minimal (by (69)). Therefore, by a result of [24], $S_{f,n}$ has to be invertible (*i.e.* $n = \pm 1$). Denote

$$C_\infty(T_\varphi) = \{S \in C(T)\colon (\exists \tilde{f}\colon \mathbf{R} \longrightarrow \mathbf{R} \text{ periodic of period 1}, \text{ of } C^\infty-\text{class})\ \ S_{\exp(2\pi i\tilde{f}),1} \in C(T_\varphi)\}.$$

PROPOSITION 4: *If an ergodic Anzai skew product T_φ is given by $\varphi = \exp(2\pi i\tilde{\varphi})$, where $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is periodic of period 1, of C^∞-class and $\int_0^1 \tilde{\varphi}(t)dt = 0$, then $C_\infty(T_\varphi)$ is uncountable.*

We will need some auxiliary results. For $t \in \mathbf{R}$ denote $\|t\| = \min_{n\in\mathbf{Z}} |t-n|$. For a continuous function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ periodic of period 1, we put $\|f\| = \sup_{t\in[0,1]} |f(t)|$.

LEMMA 4 (([3], [9])): *Let $\alpha \in [0,1)$ be irrational with the sequence $\{q_n\}$ of denominators of α. Let $\tilde{\psi}\colon [0,1) \longrightarrow \mathbf{R}$, $\int_0^1 \tilde{\psi}(x)dx = 0$, $\tilde{\psi}(0) = \tilde{\psi}(1)$ and $\tilde{\psi}$ be absolutely continuous. Then $\{\tilde{\psi}^{(q_n)}\}$ tends to zero uniformly on $[0,1]$.*

LEMMA 5: *Suppose that $\tilde{\varphi}\colon \mathbf{R} \longrightarrow \mathbf{R}$ is periodic of period 1, $\int_0^1 \tilde{\varphi}(t)dt = 0$ and $\tilde{\varphi} \in C^m(\mathbf{R})$ with $\frac{d^m\tilde{\varphi}(x)}{dx^m}$ absolutely continuous. Then, the set*

$$C_m(T_\varphi) = \{S \in C(T)\colon (\exists \tilde{f}\colon \mathbf{R} \longrightarrow \mathbf{R} \text{ periodic of period 1 of } C^m-\text{class})\ \ S_{\exp(2\pi i\tilde{f}),1} \in C(T_\varphi)\}$$

is uncountable.

Proof: Assume that $Tx = x \cdot e^{2\pi i\alpha}, x \in X$ and let $\{q_n\}$ be the sequence of denominators of α. Then $\lim_{n\to\infty} ||q_n\alpha|| = 0$. By chooosing a subsequence of $\{q_n\}$, if necessary, we can assume that

$$||q_n\alpha|| > \sum_{k=n+1}^{\infty} ||q_k\alpha||. \tag{71}$$

Denote $\tilde{\psi}_p = \frac{d^p\tilde{\varphi}(x)}{dx^p}, p = 0,1,\ldots,m$. Then $\int_0^1 \tilde{\psi}_p(x)dx = \frac{d^{p-1}\tilde{\varphi}(1)}{dx^{p-1}} - \frac{d^{p-1}\tilde{\varphi}(0)}{dx^{p-1}} = 0$, and $\tilde{\psi}$ is absolutely continuous. In view of Lemma 4, $\tilde{\psi}_p^{(q_n)} \longrightarrow 0$ uniformly on $\mathbf{R}$. By choosing a subsequence of $\{q_n\}$, if necessary, we can assume that

$$||\tilde{\psi}_p^{(q_n)}|| \geq \sum_{k=n+1}^{\infty} ||\tilde{\psi}_p^{(q_k)}|| \tag{72}$$

for n large enough and $p = 0,1,\ldots,m$. Let $r = (r_0, r_1, \ldots)$ be any sequence with $r_i = 0,1,\ i \geq 1$. Denote

$$\alpha_{n,r} = \sum_{k=0}^{n-1} r_k q_k \alpha \pmod 1, \quad \tilde{\psi}_{n,r,p} = \tilde{\psi}_p^{(\sum_{k=0}^{n-1} r_k q_k)}.$$

Now, by (71), $\{(\alpha_{n,r})\colon n \geq 0\}$ is a Cauchy sequence (mod 1) and by (72), $\{(\tilde{\psi}_{n,r,p})\colon n \geq 0\}$ satisfies the uniform Cauchy condition. Denote

$$\beta_r = \lim_n \alpha_{n,r}, \pmod 1,$$

$$\tilde{g}_{r,p} = \lim_n \tilde{\psi}_{n,r,p} \ (\text{in } C(\mathbf{R})), \quad p = 0,1,\ldots,m,$$

$$\tilde{f}_r = \lim_n \tilde{\varphi}^{(\sum_{k=0}^{n-1} r_k q_k)} \ (\text{in } C(\mathbf{R})).$$

We have

$$\frac{d^p\tilde{f}_r(x)}{dx^p} = \tilde{g}_{r,p}(x), \quad p = 0, 1, \ldots, m.$$

Note that if $r \neq r'$ then $\beta_r \neq \beta_{r'}$. Indeed, let i be the smallest number such that $r_i \neq r'_i$. Then

$$||\beta_r - \beta_{r'}|| \geq ||q_i\alpha|| - \sum_{t=i+1}^{\infty} ||q_t\alpha|| > 0.$$

Let $S_r x = x \cdot e^{2\pi i\beta_r}$. It remains to prove that $(S_r)_{\exp(2\pi i\tilde{f}_r),1} \in C(T_\varphi)$. But, we have

$$\begin{aligned} &|(T_\varphi)^{\sum_{k=0}^{n-1} r_k q_k}(x,y) - (S_r)_{\exp(2\pi i\tilde{f}),1}(x,y)| \\ &= |(\exp(2\pi i\alpha_{n,r}) \cdot x, \tilde{\varphi}^{(\sum_{k=0}^{n-1} r_k q_k)}(x) \cdot y) - (\exp(2\pi i \cdot \beta_r) \cdot x, \exp(2\pi i\tilde{f}_r(x)) \cdot y)| \\ &\leq ||\alpha_{n,r} - \beta_r|| + ||\tilde{\varphi}^{(\sum_{k=0}^{n-1} r_k q_k)} - \tilde{f}_r|| \longrightarrow 0. \end{aligned}$$

Therefore, $(S_r)_{\exp(2\pi i\tilde{f}),1}$ commutes with T_φ and the result follows. ∎

Proof of Proposition 4: This is a small modification of the proof of Lemma 5. We have to choose a subsequence $\{q_{n_k}\}$ of $\{q_n\}$ satisfying

$$||(\frac{d^m\tilde{\varphi}(x)}{dx^m})^{(q_{n_k})}|| \geq \sum_{r=k+1}^{\infty} ||(\frac{d^m\tilde{\varphi}(x)}{dx^m})^{(q_{n_r})}||,$$

for $k \geq N_m$, $m \geq 0$.

This can be done by the standard diagonal procedure. So, the arguments needed in the proof of Lemma 5 work well and the result follows. ∎

6. Remarks

6.1 The constructions of this paper depend heavily on properties of α. We recall the result of [8] saying that if

$$\alpha = [0\colon a_1, a_2, \ldots)$$

satisfies

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{Q_n} < +\infty \tag{73}$$

then there is no nontrivial (*i.e.* not cohomologous to a constant) C^2-cocycles. The reader can see how opposite to (R1) the condition (73) is. We can slightly weaken (73) to $\sum_{n=0}^{\infty} \frac{a_{n+1}}{Q_n^{\delta}} < +\infty$ for some $1 > \delta > 0$ obtaining that each $C^{1+\delta}$-cocycle of degree zero is cohomologous to a constant (see [3]).

Notice, also that each number α satisfying (R1) has to be a Liouville number. Indeed, we have $a_{2n_k+1} \geq Q_{2n_k}^k$. Hence

$$\frac{1}{Q_{2n_k}Q_{2n_k+1}} \leq \frac{1}{a_{2n_k+1}Q_{2n_k}^2} \leq \frac{1}{Q_{2n_k}^{k+2}}$$

and by (75) (in the appendix)

$$|\alpha - \frac{P_{2n_k}}{Q_{2n_k}}| < \frac{1}{Q_{2n_k}^{k+2}}, \quad k = 1, 2, \dots .$$

In particular, the set of α's satisfying the (R) condition is of zero Hausdorff dimension.

In fact, if $\tilde{\varphi}$ is of C^∞-class and $\varphi = \exp 2\pi i \tilde{\varphi}$ is not T-cohomologous ($Tx = x + \alpha$) to a constant, then α has to be a Liouville number.

6.2 Corollary 3 is the affirmative answer to the question formally raised by J.-P. Thouvenot. Also, it gives the negative answer to Question 4 from [17]. Can Corollary 3 be strengthened to get $\tilde{\varphi}$ analytic?

6.3 In Corollary 3, we prove that there exists an ergodic Anzai skew product T_φ, where φ is of C^∞-class, such that there are two measure-preserving maps Θ, Σ: $(X \times X, \tilde{\mu}) \longrightarrow (X \times X, \tilde{\mu})$ such that

$$\Theta T_\varphi = T_{\varphi^2}\Theta, \;\; \Sigma T_{\varphi^2} = T_\varphi \Sigma.$$

The map Θ is also of C^∞-class. However, Σ is not. Actually, Σ must not be continuous whenever Θ is. Indeed, otherwise $\Theta \circ \Sigma$ is a continuous map commuting with the homeomorphism T_φ which is minimal. By a result of [24], $\Theta \circ \Sigma$ has to be invertible, so both Σ and Θ are invertible, a contradiction.

Can we find two diffeomorphisms T_1, T_2 of $\mathrm{S}^1 \times \mathrm{S}^1$ preserving Lebesgue measure $\tilde{\mu} = \mu \times \mu$, ergodic with respect to $\tilde{\mu}$, which are smooth factors of each other (*i.e.* Θ, Σ are smooth) but are not measure-theoretically isomorphic?

6.4 It is well-known ([14], [27]) that given rotation by some $e^{2\pi i\alpha}$, each cocycle $\varphi\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ is cohomologous to a continuous one.

Given a cocycle $\varphi\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$, can we find a continuous $\varphi_1\colon \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ with bounded variation such that φ_1 is cohomologous to φ?

7. Appendix

Let $\alpha \in [0,1)$ be irrational, with the continued fraction expansion

$$\alpha = [0\colon a_1, a_2, \ldots) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

The positive integers a_i are said to be the **partial quotients** of α. Put

$$\text{(74)} \qquad \begin{array}{lll} Q_0 = 1, & Q_1 = a_1, & Q_{n+1} = a_{n+1}Q_n + Q_{n-1} \\ P_0 = 0, & P_1 = 1, & P_{n+1} = a_{n+1}P_n + P_{n-1}. \end{array}$$

The rationals P_n/Q_n are called the **convergents** of α and the following

$$\text{(75)} \qquad |\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_nQ_{n+1}}$$

holds. The following formula

$$\text{(76)} \qquad Q_{n+1}||Q_n\alpha|| + Q_n||Q_{n+1}\alpha|| = 1$$

holds true (where $||t||$ denotes the distance of a real number t from the set of integers).

The result below is a direct consequence of the definition of the continued fraction expansion of α and (76).

PROPOSITION 5: *Let $n \geq 2$ be even. Then, the intervals*
$[0, Q_n\alpha), \tilde{T}[0, Q_n\alpha), \ldots, \tilde{T}^{(a_{n+1}Q_n+Q_{n-1})-1}[0, Q_n\alpha)$,
$[Q_{n+1}\alpha, 1), \ldots, \tilde{T}^{Q_n-1}[Q_{n+1}\alpha, 1)$
are pairwise disjoint with the union equal to $[0,1)$. *Moreover*
$[0, a_{n+1}Q_n\alpha) = [0, Q_n\alpha) \cup \tilde{T}^{Q_n}[0, Q_n\alpha) \cup \ldots \cup \tilde{T}^{(a_{n+1}-1)Q_n}[0, Q_n\alpha)$.

Suppose, now, that α satisfies (R2) and (R4). Then, from Proposition 1 and (R4) it follows that (15), (17) and (18) hold true. Moreover, by (i) and (ii) of the above proposition, we have

$$\mu((\bigcup_{i=0}^{h_k-1}\tilde{T}^iI_k)^c)\le Q_{2n_k-1}||Q_{2n_k}\alpha||+Q_{2n_k}||Q_{2n_k+1}\alpha||$$

so, by (75) and (74)

$$\mu((\bigcup_{i=0}^{h_k-1}\tilde{T}^iI_k)^c)\le 2Q_{2n_k}\cdot\frac{1}{Q_{2n_k+1}}<\frac{2}{a_{2n_k+1}}.$$

Therefore, (16) follows.

Proof of Theorem 1: Denote

$$X_{R1,s}=\{\alpha\in[0,1)\colon \frac{Q_s^t}{a_{s+1}}C_t<2^{-s}\ \text{ for }\ t=1,\dots,s\},$$

$$X_{R2,s}=\{\alpha\in[0,1)\colon \frac{2}{a_{s+1}}<\min\{\varepsilon_1,\dots,\varepsilon_s\}\},$$

$$X_{R3,s}=\{\alpha\in[0,1)\colon a_{s+1}=p(s)q(s),\ p(s)>\max(\frac{1}{4\varepsilon_1},\dots,\frac{1}{4\varepsilon_s}),\ q(s)>s\},$$

$$X_{R4,s}=\{\alpha\in[0,1)\colon Q_s>\max(\frac{1}{\varepsilon_1^2},\dots,\frac{1}{\varepsilon_s^2})\}.$$

Now, notice that if an irrational number $\alpha\in X_{Ri,s}$, $\varepsilon>0$ is small enough that each irrational β, $||\beta-\alpha||<\varepsilon$ must belong to $X_{Ri,s}$. In other words, there exists an open set $X_{Ri,s}^{\circ}\subset[0,1)$ such that $X_{Ri,s}=X_{Ri,s}^{\circ}\cap([0,1)\setminus\mathbf{Q})$. Now, put

$$A=\bigcap_{t=1}^{\infty}\bigcup_{s=t}^{\infty}X_{R1,2s}^{\circ}\cap X_{R2,2s}^{\circ}\cap X_{R3,2s}^{\circ}\cap X_{R4,2s}^{\circ}.$$

Notice that $\bigcup_{s=t}^{\infty}X_{R1,2s}^{\circ}\cap X_{R2,2s}^{\circ}\cap X_{R3,2s}^{\circ}\cap X_{R4,2s}^{\circ}$ is dense and open. Moreover, if $\alpha\in A$ then certainly it satisfies the full (R) condition and the proof is complete. ∎

References

[1] L.M. Abramov, *Metric automorphisms with quasi-discrete spectrum*, Izw. AN SSSR **26** (1962), 513–550 (in Russian).

[2] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka J. Math. **3** (1951), 83–99.

[3] P. Gabriel, M. Lemańczyk and P. Liardet, *Ensemble d'invariants pour les produits croisés de Anzai*, Mémoire SMF no. 47, tome 119 (3) (1991) (in French).

[4] P. Gabriel, M.L. Lemańczyk and M.K. Mentzen, *Two-point cocycles with a strong ergodicity property*, Bull. Pol. Ac. Sc. 37 no 1-6 (1989), 355–362.

[5] W.H. Gottschalk and G.A. Hedlund, *Topological Dynamics*, Amer. Soc. Colloq. Publ. vol. 36, 1955.

[6] F. Hahn and W. Parry, *Some characteristic properties of dynamical systems with quasi-discrete spectrum*, Math. Syst. Th. **2** (1968), 179–198.

[7] P.R. Halmos, *Lectures on Ergodic Theory*, Chelsea Publ. Co., New York, 1956.

[8] P.Hellekalek and G.Larcher, *On Weyl sums and skew products over irrational rotations*, Th. Comp. Sc., Fund. St. 62, 2 (1989), 189-196.

[9] M. Herman, *Sur la conjugaison differentiable des difféomorphismes du cercle à des rotation*, Publ. I.H.E.S. 1976.

[10] A. del Junco, *A family of counterexamples in ergodic theory*, Isr. J. Math. **44** (1983), 160–188.

[11] A. del Junco and M. Lemańczyk, *Generic spectral properties of measure-preserving maps, and applications*, to appear in Proc. Amer. Math. Soc. (1991).

[12] A. del Junco, A. Rahe and L. Swanson, *Chacon's automorphism has minimal self-joinings*, J. Analyse Math. **37** (1980), 276–284.

[13] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Erg. Th. Dyn. Syst. **7** (1987), 531–557.

[14] A.W. Kočergin, *On the homology of functions over dynamical systems*, Dokl. AN SSSR **281** (1976).

[15] M. Lemańczyk, *Weakly isomorphic transformations that are not isomorphic*, Prob. Th. Rel. Fields **78** (1988), 491–507.

[16] M. Lemańczyk, *On the weak isomorphism of strictly ergodic homeomorphisms*, Monatshefte Math. **108** (1989), 39–46.

[17] M. Lemańczyk and P. Liardet, *Coalescence of Anzai skew products*, unpublished preprint.

[18] D. Newton, *Coalescence and spectrum of automorphisms of a Lebesgue space*, Z. Wahr. Verw. Geb. **19** (1971), 117–122.

[19] D. Newton, *On canonical factors of ergodic dynamical systems*, J. London Math. Soc. (2) **19** (1979), 129–136.

[20] D. Ornstein, *Bernoulli shifts with the same entropy are isomomorphic*, Adv. Math. **4** (1970), 337–359.

[21] D. Ornstein, *Factors of Bernoulli shifts are Bernoulli*, Adv. Math. **5** (1970), 349–364.

[22] D. Ornstein, D. Rudolph and B. Weiss, *Equivalence of measure-preserving transformations*, Memoirs of AMS **37** no 262 (1982).

[23] W. Parry, *Compact abelian group extensions of discrete dynamical systems*, Z. Wahr. Verw. Geb. **19** (1969), 95–113.

[24] W. Parry and P. Walters, *Minimal skew-product homeomorphisms and coalescence*, Compositio Math. (3) **22** (1970), 283–288.

[25] S. Polit, *Weakly isomorphic transformations need not be isomorphic*, Thesis, Stanford Univ., 1975.

[26] D. Rudolph, *An example of a measure-preserving map with minimal self-joinings and applications*, J. Analyse Math. **35** (1979), 97–122.

[27] D. Rudolph, $\mathbf{Z}^n$ *and* $\mathbf{R}^n$ *cocycle extensions and complementary algebras*, Erg. Th. Dyn. Syst. **6** (1986), 583–599.

[28] D. Rudolph, *Inner and barely linear time changes of ergodic* $\mathbf{R}^n$*-actions*, Conf. in Mod. Anal. and Prob., Contemp. Math (Kakutani conf.) **26**, (1983), 351–372.

[29] Ya.G. Sinai, *Weak isomorphism of transformations with invariant measure*, Mat. Sb. **63** (1963), 23–42 (in Russian).

[30] J.-P. Thouvenot, *The metrical structure of some Gaussian processes*, Proc. Erg. Th. Rel. Topics II, Georgenthal 1986, pp. 195–198.